Beam stability in synchrotron light sources

Everything you need to know about accelerator physics to get the job done - a seat of the pants approach.

Particle trajectories in a storage ring satisfy a differential equation called Hill's equation, which I won't bother to write down. Rather, all you really need to know boils down to the following generic solutions to Hill's equation:

$$x(s) = \mathrm{sqrt}[W_x\ \beta_x(s)] * \cos[\ \phi_x(s) - \phi_0] \qquad (1)$$

$$y(s) = \mathrm{sqrt}[W_y\ \beta_y(s)] * \cos[\ \phi_y(s) - \phi_0] \qquad (2)$$

This equation lives in the curvilinear coordinate system (x,y,s). The variable s is the path length along the equilibrium closed orbit. The function y(s) is the vertical displacement of a particular particle's trajectory relative to the equilibrium closed orbit, and x(s) is the radial displacement. The variables (x,y,s) form a right-handed triad, which means that any sensible accelerator has the particle beam moving clockwise. In Japan, for example, they drive on the left side of the road, and similarly their big light source SPring-8 has the particle beam going counterclockwise. So at the clockwise Advanced Photon Source (APS), for example, x(s) is positive if the closed orbit is displaced radially outboard from the equilibrium closed orbit. Also, In the vertical plane, positive y means up.

The term "equilibrium closed orbit" is somewhat of a misnomer. In fact, any trajectory you can get which provides a stable stored beam can be declared to be "the" equilibrium closed orbit. If you then disturb the beam, e.g. by changing the magnetic field in a steering corrector magnet, the difference between the two stable closed orbits turns out to take the same form as equation 1, but with the additional periodic boundary condition x(s) = x(s+L) imposed upon it, where L is the storage ring circumference. A kink will appear in x(s) at the value of s corresponding to the corrector location, i.e. the first derivative x'(s) is discontinuous there. This periodic solution is a property of an ensemble of particles and parametrizes the center of mass of a bunch of charged particles.
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Any individual particle's trajectory satisfies equation (1), but is explicitly not periodic, in other words the tune ν is an irrational number. The tune (horizontal or vertical) is just the betatron phase advance per turn, divided by 2 π:

$$\nu = [\phi(s+L) - \phi(s)] / 2\ \pi \qquad (3)$$

Note that the beta function $\beta(s)$ is periodic, $\beta(s) = \beta(s+L)$, while the betatron phase $\phi(s)$ is a monotonically increasing function of s. Synchrotron light sources tend to have a relatively large horizontal tune ν_x in comparison to older storage rings and synchrotons, which used things like FODO lattices. I think of this high tune value as "pinching it until it squeaks" -- to obtain very small transverse particle beam size, you need a low emittance, which generally means that you end up with a relatively large value of ν_x, and correspondingly large amounts of quadrupole magnet focussing - the "pinch". More precisely, it is the betatron phase advance per superperiod that gets large in low-emittance machines. This is just the tune divided by the number of superperiods. At the APS, the storage ring has forty-fold symmetry, i.e. it has forty superperiods, and the horizontal tune is 36 plus change. That's big. Spring-8 has 48 superperiods and ESRF has 32. Emittance decreases as the cube of the number of superperiods, which is why more is better, until you run out of money.

The tunes ν_x or ν_y as written in equation (3) aren't obviously independent of the azimuthal variable s, however it turns out that the betatron phase $\phi(s)$ depends on the periodic beta function thusly:

$$\phi(s) = \phi_0 + \int ds \, (1 / \beta(s)) \qquad (4)$$

If you stare at this long enough, you should be able to convince yourself that the tune of equation (3) will give the same result when integrated starting from any value of s to s+L.

Lattice designers generally will display plots of β_x and β_y as a function of s for one superperiod. Figure 1 shows the lattice functions for a recent incarnation of the APS lattice.

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Notice that in general β_x has a local maximum value where β_y is minimum, and vice-versa. This has to do with the curl equation of Maxwell fame, which dictates that quadrupole magnets that focus horizontally defocus vertically, and vice versa.

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Also shown in Figure 1 is a third function declared to be D(s) in Europe and η (s) in North America. Worldwide, I think everyone calls it "the dispersion". The dispersion is the most direct way to express what happens to off-energy particles. Generally particles in a storage ring used for synchrotron radiation have an energy spread of about 0.1% rms. Higher energy particles tend to get bent less by the bending magnets and focussed less by the focussing quadrupoles. What this tends to mean is that higher energy particles on average follow a path with a larger circumference than low energy particles (I'm ignoring bizarre things like isochronous lattices and negative momentum compaction lattices). The higher energy particles take longer to make a round trip (remember, they're all moving at essentially the speed of light), and this turns out to be really important. In addition to the longer round trip period, the higher energy particles are displaced radially relative to on-energy particles by an amount

$$\Delta x \, (s) = \eta(s) \, (\Delta E / E) \, . \qquad (5)$$

Note that there are places around the ring where the dispersion is zero, which means that there is no correlation between the radial position and energy of the ensemble of particles making up a bunch. In general there is such a thing as vertical dispersion, but this is usually much smaller than for the horizontal plane. The vertical dispersion is exactly zero if the equilibrium closed orbit lies exactly in a horizontal plane.

Emittance vs. The Courant-Snyder Invariant

The constant W_x in equations 1 is called the Courant-Snyder invariant. It reflects the amplitude of a betatron oscillation exhibited by a single particle. It is very similar to the energy associated with a harmonic oscillator. In fact, W_x appears to be a quadratic function of x(s) and it's first derivative x'(s).

$$W_x = \gamma x^2 + 2 \alpha x x' + \beta x'^2 \qquad (6)$$

The three lattice functions $\gamma(s)$, $\alpha(s)$, and $\beta(s)$ are called Twiss parameters. They are a convenient way to parametrize phase space ellipses in storage rings. The invariant W_x is proportional to the area of the ellipse in [x, x'] phase space. For a fixed location s along the equilibrium closed orbit on successive turns n, the ordered pairs [x(s+nL), x'(s+nL)] sample different points on the phase space ellipse given by equation (6). If you wait long enough, the given particle will sample every point along the ellipse, owing to the fact that the tune is an irrational number.

Quite often one will see equations like (1) and especially (6) using the symbol ε_x in place of W_x . The former nomenclature tends to elicit the idea of “emittance”, which is a distinctively different concept from Courant-Snyder invariant. Emittance is a statistical property of an ensemble of many particles, while the Courant-Snyder invariant is a parameter associated with a single particle. In the context of equation (6), the ellipse corresponding to $W_x = \varepsilon_x$ represents the ellipse within which 68% of all particles' phase space trajectories lie.